VIII. "On the Occlusion of Hydrogen and Oxygen by Palladium." By LUDWIG MOND, Ph.D., F.R.S., WILLIAM RAMSAY, Ph.D., LL.D., Sc.D., F.R.S., and JOHN SHIELDS, D.Sc., Ph.D.

The Society adjourned over the Christmas Recess to Thursday, January 20, 1898.

"On a Method of determining the Reactions at the Points of Support of Continuous Beams." By GEORGE WILSON, M.Sc., Demonstrator in Engineering in the Whitworth Laboratory of the Owens College, Manchester. Communicated by Professor OSBORNE REYNOLDS, F.R.S. Received November 20,—Read December 16, 1897.

The theory of continuous beams has been the subject of so much research in the past that further investigation would seem almost superfluous. In certain cases which occur in practice, however, the computations arising out of the existing methods become complicated and laborious, if not impossible to reduce, so that any solution which avoids these difficulties may be of sufficient value to warrant its publication.

Mr. Heppel, in a paper read before this Society,* has traced the developments in the theory, culminating in the discovery of the 'Theorem of Three Moments,' by M. Bertot, in 1856, and independently by M.M. Clapeyron and Bresse, in 1857. Previously to Clapeyron, Navier and other authors had sought the solution of the problem by obtaining the reactions at the various points of support of the beam; Clapeyron, however, first introduced the innovation of considering the bending moments at the points of support as the unknown quantities to be determined.

M. Bresse, in his 'Cours de Mécanique Appliquée,' has discussed very fully the solutions of the various problems by this method, on the supposition of a constant moment of inflexibility of the sections of the beam both for the case of spans of arbitrary lengths and also for cases where the end spars are equal but of different length to the intermediate spans whose lengths are all supposed to be equal.

Mr. Heppel, in the above mentioned paper, published solutions in which the spans were divided into two, three, four, or five equal parts throughout each of which the load and the cross section of the beam were supposed to remain constant, although varying from one division to another.

Professors Perry and Ayrton† have dealt with the question of a

* 'Roy. Soc. Proc.,' vol. 18, p. 176.

† *Ibid.*, vol. 29, p. 493.

variable moment of inertia by obtaining the theorem of three moments in a slightly different form, the necessary summations for each span being performed graphically, whence on substitution in the original equations the bending moments can be obtained. The author has reverted to the problem of finding the reactions at the points of support and has based his method on a principle, definitely stated by Bresse,* and applied by him to the case of a uniform continuous beam of two equal spans.

The author claims that the method given below affords an easy and accurate solution for continuous beam problems, and especially those in which the moment of inertia is variable. It also permits of the variations in the stresses due to alterations in the levels of the supports being investigated.

The principle may be reproduced as far as is necessary as follows:—

The displacement of any point by reason of the deformation of the beam is the resultant of the displacements which would be produced if one supposed all the external known forces to act separately and one after the other.

This being so, the continuous beam may be considered as a simple beam supported at each end and under the action of the given loading acting vertically downwards, and also under the action of the supporting forces at the intermediate piers acting vertically upwards.

If the neutral fibre of the beam in the unloaded condition is assumed to be a straight line, then the result of the action of these two distinct systems of loading is to make the final deflection of the neutral fibre at each of the intermediate points of support equal to zero.

If the beam consist of n spans there will be $n-1$ intermediate supports, the upward pressure of each of which acting by itself would produce a definite deflection of the beam at any point: the sum of the separate deflections produced by these pressures at any one point of support must equal the deflection produced there by the given loading, and as each of the constituent deflections can be expressed in terms of the unknown concentrated load causing it, there will, therefore, be $(n-1)$ equations each containing the reactions at each intermediate point of support, and as there are $(n-1)$ reactions these equations are sufficient to determine their values.

Let $A_0\ A_1\ A_2 \ldots\ A_n$ be the points of support.

$$A_0A_n = L.$$

$$A_0A_1 = l_1, \qquad A_0A_2 = l_2, \qquad A_0A_3 = l_3 \ldots .$$

* 'Cours de Mécanique Appliquée,' vol. 1, p. 137.

Let m be the bending moment at any point in the mean fibre of the beam, due to the given system of loading.

I = the moment of inertia of the section of the beam upon which m acts.

E = the modulus of elasticity.

Let the origin be at A_0; the axis of x be $A_0A_1 \ldots A_n$; and the axis of y be perpendicular to that of x and positive downwards.

Also let the suffixes 0, 1, 2, 3,.... refer to the corresponding points $A_0, A_1, A_2, A_3, \ldots$ so that m_1 is the bending moment at the first intermediate point of support, due to the given loading.

Then from the equation

$$\mathrm{E}\frac{d^2y}{dx^2} = -\frac{m}{\mathrm{I}}$$

for the case of a beam supported at each end, we obtain

$$\mathrm{E}y = -\iint\frac{m}{\mathrm{I}}\,dx^2 + \mathrm{ET}_0x,$$

where T_0 is the tangent of the angle of inclination to the axis of x, of the tangent to the mean fibre at the origin—

$$\therefore\ \mathrm{E}y_1 = -\iint_0^{l_1}\frac{m}{\mathrm{I}}\,dx^2 + \mathrm{ET}_0l_1,$$

where y_1 is the deflection at A_1 which would be produced were the beam only supported at each end and under the action of the given loading.

Again let m', m'', m''', &c., be the bending moments at any point in the mean fibre due to the upward thrust of the reactions R_1, R_2, $R_3 \ldots$ at the points $A_1, A_2, A_3 \ldots$, and T_0', T_0'', T_0''' the corresponding tangents at the origin.

Then
$$\mathrm{E}y_1' = -\iint_0^{l_1}\frac{m'}{\mathrm{I}}\,dx^2 + \mathrm{ET}_0'l_1$$

$$\mathrm{E}y_1'' = -\iint_0^{l_1}\frac{m''}{\mathrm{I}}\,dx^2 + \mathrm{ET}_0''l_1$$

hence finally since

$$y = y' + y'' + y''' + \ldots \text{ for the points } A_1, A_2, A_3 \ldots,$$

we have

$$\mathrm{ET}_0l_1 - \iint_0^{l_1}\frac{m}{\mathrm{I}}\,dx^2 = \left(\mathrm{ET}_0'l_1 - \iint_0^{l_1}\frac{m'}{\mathrm{I}}\,dx^2\right) + \left(\mathrm{ET}_0''l_1 - \iint_0^{l_1}\frac{m''}{\mathrm{I}}\,dx^2\right) + \ldots,$$

$$\mathrm{ET}_0l_2 - \iint_0^{l_2}\frac{m}{\mathrm{I}}\,dx^2 = \left(\mathrm{ET}_0'l_2 - \iint_0^{l_2}\frac{m'}{\mathrm{I}}\,dx^2\right) + \left(\mathrm{ET}_0''l_2 - \iint_0^{l_2}\frac{m''}{\mathrm{I}}\,dx^2\right) + \ldots,$$

and so on, there being as many equations as intermediate points of support.

If the function m/I is assumed to represent the intensity of a new loading on the girder, it can easily be shown that the expression

$$\mathrm{ET}_0 l_1 - \iint_0^{l_1} \frac{m}{\mathrm{I}}\, dx^2$$

represents the value of the bending moment at the point A_1, due to this new loading, considering the beam as supported only at each end.

Let this expression for the bending moment at A_1 be called N_1, and that at A_2 N_2, and similarly for A_3 .....

Again, we may write the expression

$$\left(\mathrm{ET}_0' l_1 - \iint_0^{l_1} \frac{m'}{\mathrm{I}}\, dx^2\right) \text{ equal to } \mathrm{R}_1 n_1',$$

n_1' being the bending moment at A_1, due to a new "m/I" loading obtained by assuming a unit force to act at A_1.

Similarly $$\left(\mathrm{ET}_0'' l_1 - \iint_0^{l_1} \frac{m''}{\mathrm{I}}\, dx^2\right) = \mathrm{R}_2 n_1''.$$

Hence the equations become

$$\mathrm{N}_1 = \mathrm{R}_1 n_1' + \mathrm{R}_2 n_1'' + \mathrm{R}_3 n_1''' + \ldots\ldots,$$

$$\mathrm{N}_2 = \mathrm{R}_1 n_2' + \mathrm{R}_2 n_2'' + \mathrm{R}_3 n_2''' + \ldots\ldots,$$

$$\mathrm{N}_3 = \mathrm{R}_1 n_3' + \mathrm{R}_2 n_3'' + \mathrm{R}_3 n_3''' + \ldots\ldots,$$

&c., &c.,

from which R_1, R_2, R_3 .... may be easily obtained when the constants have been determined.

Lord Rayleigh has shown, in his 'Theory of Sound' (vol. 1, p. 69), that when a beam is loaded with a concentrated load at any point P_1, and the load is transferred to a second point P_2, then the deflection at P_2 when the load is at P_1 is equal to the deflection at P_1 when the load is at P_2, hence

$$n_2' = n_1''; \quad n_3' = n_1'''; \quad n_2''' = n_3''; \quad \text{\&c., \&c.,}$$

thus reducing the number of constants to be found, or affording a check on the accuracy of the working.

In the example appended

$$n_2' = 22{\cdot}47 \text{ by calculation.}$$

$$n_1'' = 22{\cdot}53 \quad ,,$$

$$\text{Mean} = 22{\cdot}50$$

$$\text{Error} = 0{\cdot}13 \text{ per cent.}$$

In practice it is usually very difficult to obtain the value of the $\iint_0 \frac{m}{I} dx^2$ by integration unless I is assumed to be constant, or some simple function of x.

The following method will, however, give the values of N, n', n''.... required to any degree of accuracy.

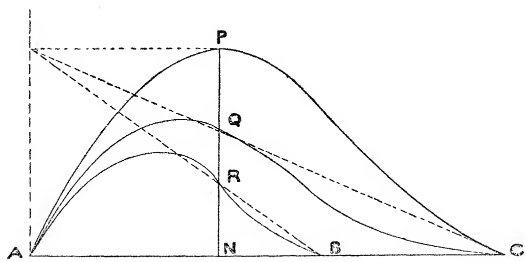


FIG. 1.

Let the diagram APCA (fig. 1), obtained by erecting ordinates proportional to the values of m/I at every point in the mean fibre AC, represent the new loading.

The bending moment at any point B may be found very simply as follows:—

If PN be any ordinate such that PN$\cdot dx$ represents the load on any small element dx at N, then if PN be divided in Q so that QN/PQ = NC/NA, then QNdx represents that part of the reaction at A due to this load PNdx at N.

Hence by taking a sufficient number of points a curve AQC can be drawn to represent the reaction at A due to the load APC. Again, the load on AB may be replaced by a load at A having an equal moment about B, in a similar manner, either graphically as in the figure, which explains itself, or by calculation. Thus a second curve, ARB, is obtained.

Then it is easily seen that the moment at B is the difference of the areas of the diagrams AQCA and ARBA, multiplied by AB, that is—

$$N_1 = AB(\text{area AQCBRA}).$$

This area may be obtained by a planimeter or by calculation, for

if all the ordinates to the curves are taken at equal intervals, the span being divided into an even number of the latter, the final diagram AQCBRA may be plotted and its area obtained by any planimeter, or by Simpson's rule.

In the example appended the area has been found both ways.

Elevation or Depression of Points of Support.

Let δ with the correct suffix represent the elevation of any point of support above the line A_0A_n, a depression being reckoned as negative.

Then in the equations to find the reactions, since the final deflection is not zero, we must write

$$N_1 + E\delta_1 = R_1 n_1' + R_2 n_1'' + \ldots.$$

$$N_2 + E\delta_2 = R_1 n_2' + R_2 n_2'' + \ldots. .$$

Some of the values of $R_1 R_2 \ldots.$ may be negative, in which case, if the beam is not to be fastened down at the supports, a fresh solution must be sought by omitting one or more of the negative reactions, until the remaining ones become positive.

The mean fibre of the beam has hitherto been assumed a straight line when under the action of no force. In certain girders this is not the case, but the above methods may be applied with sufficient accuracy for practical purposes when the maximum distance of the external layer of the beam from the mean fibre is small compared with the original radius of curvature of the mean fibre.

For then $$\frac{m}{EI} = \left(\frac{1}{R'} - \frac{1}{R}\right) \text{ approximately,}$$

where R is the original radius of curvature of the mean fibre at any point and R′ its curvature after loading.

Hence, if $v = F(x)$ is the original equation to the mean fibre, and $y = f(x)$ the equation after straining,

then $$\frac{m}{EI} = \frac{d^2y}{dx^2} - \frac{d^2v}{dx^2}$$

or $$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} + \frac{m}{EI},$$

which shows that the final deflection curve is the result of superposing on the original curve of the mean fibre, the deflection curve obtained under the given loading, for a beam of the same cross

section at every point, but with a straight mean fibre, and hence the method is applicable.

Appendix.

To find the magnitude of the reactions at the points of support of a continuous girder of three spans (viz., 60 feet, 100 feet, and 40 feet respectively), and loaded with a uniform load of 3 tons per foot run, the moment of inertia of the cross section being 2100 at the commencement of the 60-foot span and increasing uniformly to 3300 at the commencement of the 100-foot span, and thence diminishing uniformly to 3000 at the beginning of the 40-foot span, and further diminishing uniformly to 2200 at the other end of the girder, the units taken being feet and tons.

The variation of the moment of inertia is shown in fig. 2. The curve A_0BCA_3 in fig. 3 is the M/I curve for the beam resting on each end support, and loaded with 3 tons per foot run.

Treating this as a new load and reducing the ordinates in the correct proportion, the curve A_0DEA_3 shows the amount of this load transmitted to A_0 to form the reaction there.

FIG. 2.

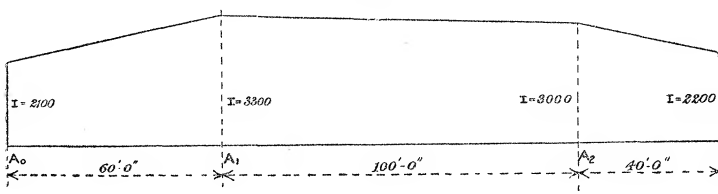
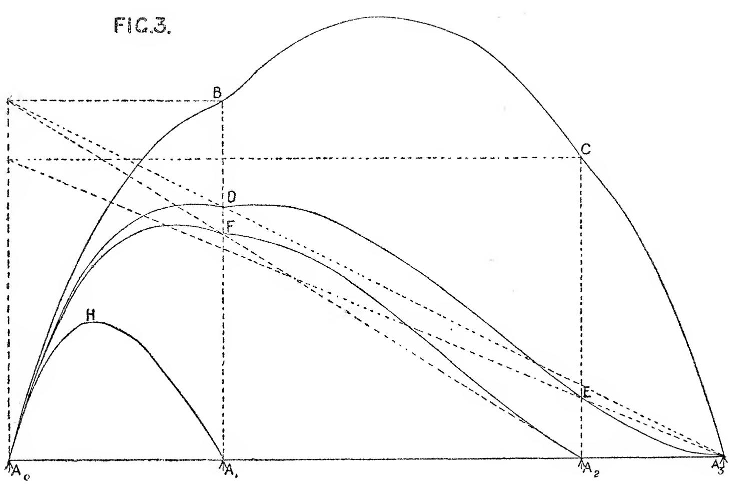


FIG. 3.



Again the curves A_0HA_1 and A_0FA_2 show the loads acting at A_0, which have their moments about A_1 and A_2 respectively, equal to the moments of the loads on A_0A_1 and A_0A_2 about A_1 and A_2 respectively.

Hence $$N_1 = \text{area } A_0DEA_3A_1HA_0 \times 60 = 16330.$$

$$N_2 = \text{area } A_0DEA_3A_2FA_0 \times 160 = 12084.$$

Similarly the curve A_0BCA_3 in fig. 4 is the M/I curves for 100-ton load acting at A_1, whence

$$n_1' = \text{area } A_0DEA_3A_1HA_0 \times \frac{60}{100} = 37{\cdot}67.$$

$$n_2' = \text{area } A_0DEA_3A_2FA_0 \times \frac{160}{100} = 22{\cdot}47.$$

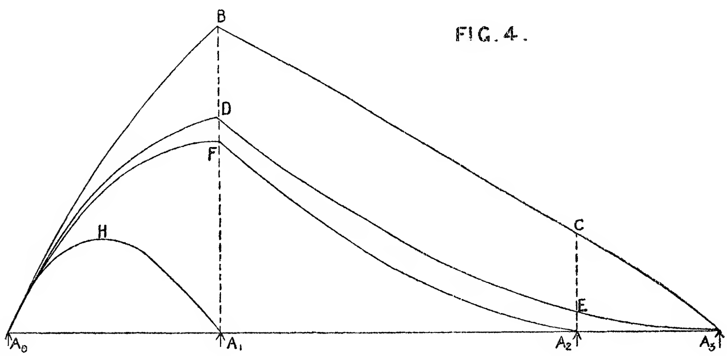


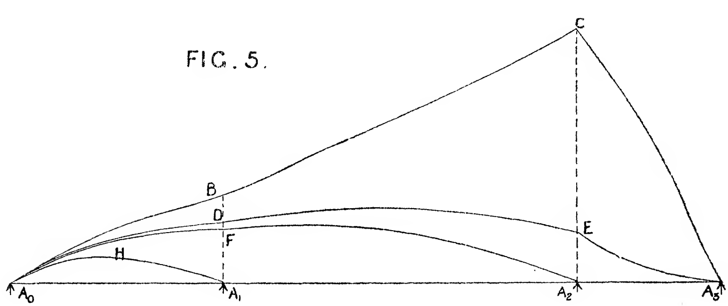
FIG. 4.

Again the curve A_0BCA_3 in fig. 5 shows the M/I curve for 100 tons load acting at A_2.

whence $$n_1'' = \text{area } A_0DEA_3A_1HA_0 \times \frac{60}{100} = 22{\cdot}53.$$

$$n_2'' = \text{area } A_0DEA_3A_2FA_0 \times \frac{60}{100} = 22{\cdot}69.$$

FIG. 5.



The equations for the supports are

$$16330 = 37{\cdot}67\ R_1 + 22{\cdot}53\ R_2,$$
$$12084 = 22{\cdot}47\ R_1 + 22{\cdot}69\ R_2,$$

whence

$$\begin{aligned} R_0 &= 52{\cdot}0 \text{ tons.} \\ R_1 &= 281{\cdot}9 \quad ,, \\ R_2 &= 253{\cdot}5 \quad ,, \\ R_3 &= 12{\cdot}6 \quad ,, \end{aligned}$$

If I were assumed to be constant then by Clapeyron's theorem of three moments

$$\begin{aligned} R_0 &= 53{\cdot}2 \text{ tons.} \\ R_1 &= 278{\cdot}3 \quad ,, \\ R_2 &= 259{\cdot}8 \quad ,, \\ R_3 &= 8{\cdot}7 \quad ,, \end{aligned}$$

The above values of the constants were obtained by dividing the span into twenty equal divisions of 10 feet, and calculating the value of the ordinates to each curve at these points. This is clearly shown in Tables I, II, III.

Having obtained the ordinates to the M/I curve, in each case the succeeding ordinates could be written down almost by inspection, and employed to find the areas by Simpson's rule, the time occupied being very short.

If the support at A_1 be supposed to be depressed $\frac{1}{2}$ inch below the line A_0A_3 whilst A_2 is supposed to be elevated $\frac{3}{4}$ inch above it, the alteration in the supporting forces is easily found.

For assuming E = 24,000,000 lbs.

$$E\delta_1 = -446{\cdot}4.$$
$$E\delta_2 = +669{\cdot}6.$$

Hence the equations become

$$16330 - 446{\cdot}4 = 37{\cdot}67\ R_1 + 22{\cdot}53\ R_2,$$
$$12084 + 669{\cdot}6 = 22{\cdot}47\ R_1 + 22{\cdot}69\ R_2,$$

whence

$$\begin{aligned} R_0 &= 82{\cdot}06. \\ R_1 &= 210{\cdot}2. \\ R_2 &= 354{\cdot}0. \\ R_3 &= -46{\cdot}26, \end{aligned}$$

indicating that under these conditions the end of the girder at A_3 would have to be fastened down in some way.

Further alterations of level may be investigated in a similar manner.

Table I.

Distance of ordinate from A_0 (feet)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
Moment of inertia (feet units)	2100	2300	2500	2700	2900	3100	3300	3270	3240	3210	3180	3150	3120	3090	3060	3030	3000	2800	2600	2400	2200
Bending moment in foot tons	0	2850	5100	7650	9600	11250	12600	13650	14400	14850	15000	14850	14400	13350	12600	11250	9600	7650	5400	2850	0
Ordinates to m/I curve	0	1·239	2·160	2·833	3·310	3·629	3·818	4·174	4·444	4·626	4·717	4·714	4·615	4·418	4·117	3·712	3·200	2·732	2·077	1·187	0
Reaction at A_0 due to m/I loading..............	0	1·177	1·944	2·408	2·648	2·722	2·673	2·713	2·666	2·544	2·358	2·121	1·846	1·546	1·235	0·928	0·640	0·410	0·208	0·060	0
Load at A_0 having moment about A_1, equivalent to that of m/I loading on A_0A_1	0	1·033	1·440	1·416	1·103	0·605	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_1HA_0$ in fig. 3	0	0·144	0·504	0·992	1·545	2·117	2·673	2·713	2·666	2·544	2·358	2·121	1·846	1·546	1·235	0·928	0·640	0·410	0·203	0·060	0
Hence area by Simpson's rule =	272·166									$N_1 = 272·166 \times 60 = 16330.$											
Load at A_0 due to m/I loading	0	1·177	1·944	2·408	2·648	2·722	2·673	2·713	2·666	2·544	2·358	2·121	1·846	1·546	1·235	0·928	0·640	0·410	0·205	0·060	0
Load at A_0 having moment about A_2, equivalent to that of m/I loading on A_0A_2	0	1·162	1·890	2·302	2·483	2·495	2·386	2·348	2·222	2·024	1·769	1·473	1·154	0·828	0·515	0·232	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_2FA_0$ in fig. 3	0	0·015	0·054	0·106	0·165	0·227	0·287	0·365	0·444	0·520	0·589	0·648	0·692	0·718	0·720	0·696	0·640	0·410	0·208	0·060	0
Hence area by Simpson's rule =	75·53									$N_2 = 75·53 \times 160 = 12084.$											

Table II.

Distance of ordinate from A_0 (feet)	0	10	20	30	40	50	60	7	80	90	100	110	120	130	140	150	160	170	180	190	200
Moment of inertia (feet units)	2100	2300	2500	2700	2900	3100	3300	3270	3240	3210	3180	3150	3120	3090	3060	3030	3000	2800	2600	2400	2200
Bending moment (ft. tons) due to 100 tons at A_1	0	700	1400	2100	2800	3500	4200	3900	3600	3300	3000	2700	2400	2100	1800	1500	1200	900	600	300	0
Ordinates to m/I curve	0	0·304	0·560	0·778	0·966	1·129	1·272	1·192	1·112	1·028	0·944	0·857	0·769	0·680	0·558	0·495	0·400	0·321	0·231	0·125	0
Reaction at A_0 due to m/I loading	0	0·289	0·504	0·661	0·773	0·847	0·890	0·775	0·667	0·565	0·472	0·386	0·308	0·238	0·177	0·124	0·080	0·048	0·023	0·006	0
Load at A_0 having moment about A_1, equivalent to that of m/I loading on A_0A_1	0	0·253	0·373	0·389	0·322	0·188	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_1HA_0$ in fig. 4	0	0·036	0·131	0·272	0·451	0·659	0·890	0·775	0·667	0·565	0·472	0·386	0·308	0·238	0·177	0·124	0·080	0·048	0·023	0·006	0
Hence area by Simpson's rule =	62·783 $\therefore$ $n'_1 = \dfrac{62 \cdot 783 \times 60}{100} = 37 \cdot 67.$																				
Load at A_0 due to m/I loading	0	0·289	0·504	0·661	0·773	0·847	0·890	0·775	0·667	0·565	0·472	0·386	0·308	0·238	0·177	0·124	0·080	0·048	0·023	0·006	0
Load at A_0 having moment about A_2, equivalent to that of m/I loading on A_0A_2	0	0·285	0·490	0·632	0·724	0·776	0·795	0·671	0·556	0·450	0·354	0·268	0·192	0·127	0·074	0·031	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_2FA_0$ in fig. 4	0	0·004	0·014	0·029	0·049	0·071	0·095	0·104	0·111	0·115	0·118	0·118	0·116	0·111	0·103	0·093	0·080	0·048	0·023	0·006	0
Hence area by Simpson's rule =	14·046 $\therefore$ $n'_2 = \dfrac{14 \cdot 046 \times 160}{100} = 22 \cdot 47.$																				

Table III.

Distance of ordinates from A_0 (feet)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
Moment of inertia (feet units)	2100	2300	2500	2700	2900	3100	3300	3270	3240	3210	3180	3150	3120	3090	3060	3030	3000	2800	2600	2400	2200
Bending moment due to 100 tons at A_2	0	200	400	600	800	1000	1200	1400	1600	1800	2000	2200	2400	2600	2800	3000	3200	2400	1600	800	0
Ordinates to m/I curve	0	0·087	0·160	0·222	0·276	0·323	0·364	0·428	0·494	0·561	0·629	0·698	0·769	0·841	0·915	0·990	1·067	0·857	0·615	0·333	0
Reaction at A_0 due to m/I loading	0	0·083	0·144	0·189	0·221	0·242	0·255	0·278	0·296	0·309	0·315	0·315	0·308	0·294	0·275	0·248	0·213	0·134	0·062	0·017	0
Load at A_0 having moment about A_1, equivalent to that of m/I loading on A_0A_1	0	0·073	0·107	0·111	0·092	0·054	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_1HA_0$ in fig. 5	0	0·010	0·037	0·078	0·129	0·188	0·255	0·278	0·296	0·309	0·315	0·315	0·308	0·294	0·275	0·248	0·213	0·134	0·062	0·017	0
Hence area by Simpson's rule =	37·55 $\therefore$ $n_1'' = \frac{37\cdot55 \times 60}{100} = 22\cdot53$.																				
Load at A_0 due to m/I loading	0	0·083	0·144	0·189	0·221	0·242	0·255	0·278	0·296	0·309	0·315	0·315	0·308	0 294	0·275	0·248	0·213	0·134	0·062	0·017	0
Load at A_0 having moment about A_2, equivalent to that of m/I loading on A_0A_2	0	0·082	0·140	0·180	0·207	0·222	0·228	0·241	0·247	0·245	0·236	0·218	0·192	0·158	0·114	0·062	0	0	0	0	0
Differences = ordinates to area $A_0DEA_3A_2FA_0$ in fig. 5	0	0·001	0·004	0·009	0·014	0·020	0·027	0·037	0·049	0·064	0·079	0·097	0·116	0·136	0·161	0·186	0·213	0·134	0·062	0·017	0
Hence area by Simpson's rule =	14·18 $\therefore$ $n_2'' = \frac{14\cdot18 \times 160}{100} = 22\cdot69$.																				